Supplementary Appendix to:
"Can Volatility Solve the Naive Portfolio Puzzle?"

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The structure of this appendix is as follows. In Section S1, we present results on Sharpe ratios net of 0.5% turnover costs. First, we plot Sharpe ratios net of turnover cost relative to the naive portfolio across datasets. These plots illustrate the economic significance of employing econometric models in portfolio construction. Second, we display tables of Sharpe ratios net of turnover cost. We present pairwise comparisons between portfolio variance strategies in Section S2. Section S3 provides some implementation details regarding the econometric models we employ. We detail the portfolio models we compare in Table S4.1 of Section S4.

S1 Sharpe Ratio Net of Turnover Costs

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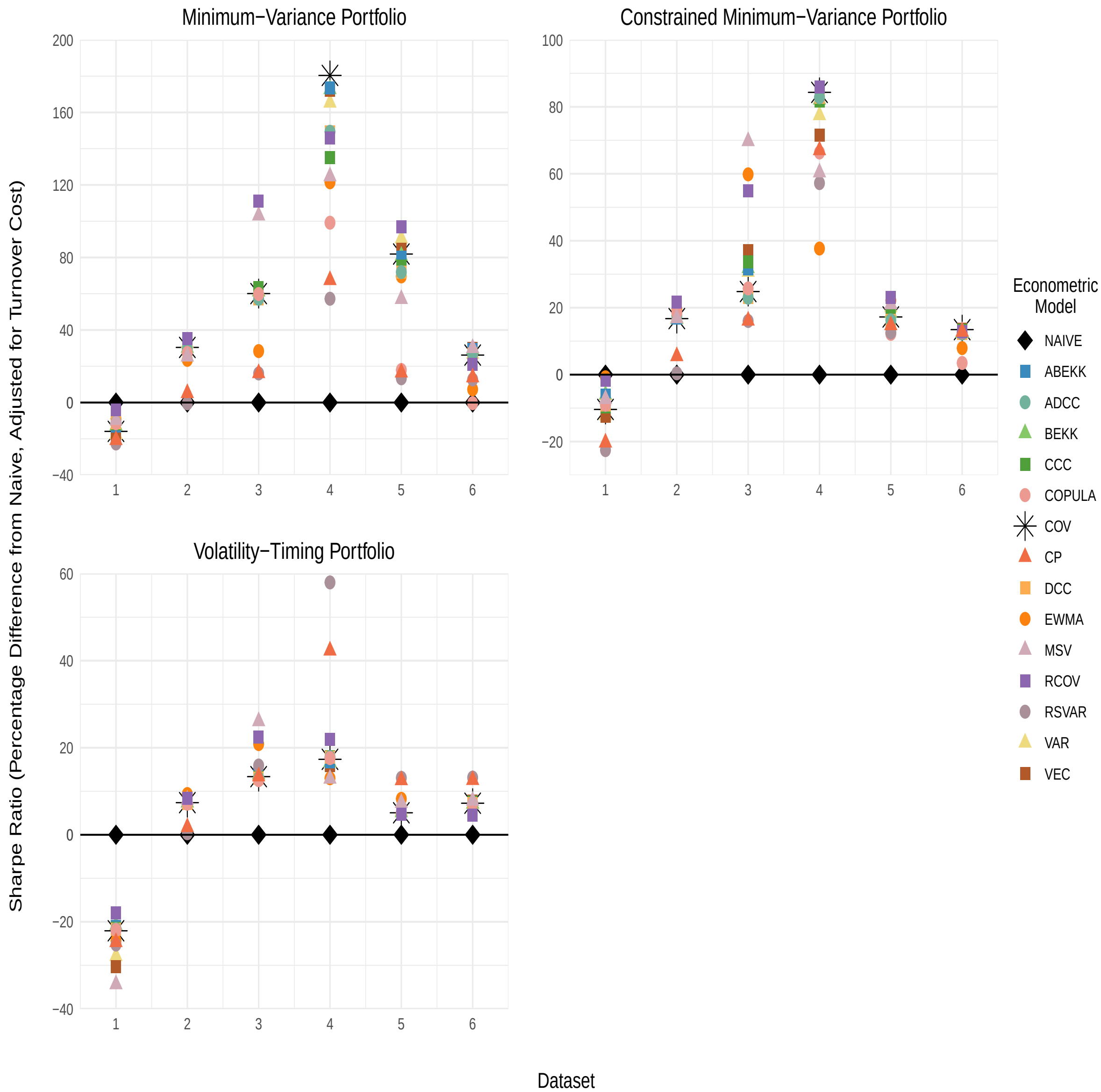
Figure S1.1: Sharpe Ratio Net of Turnover Costs Percentage Difference Relative to Naive



Notes: See Table S3.1 for explanations of econometric model abbreviations; CP denotes combined parameter model (6) from the main paper. Dataset 1: Fama-French portfolios; Dataset 2: industry portfolios; Dataset 3: sector portfolios; Dataset 4: international equity indices; Dataset 5: portfolios sorted by size/book-to-market; Dataset 6: momentum portfolios. Results are for value-weighted data at weekly frequency.

Table S1.1: Sharpe Ratio Net of Turnover Cost: Minimum-Variance and Constrained Minimum-Variance

	Dataset 1	Dataset 2	Dataset 3	Dataset 4	Dataset 5	Dataset 6
Naïve	0.113	0.101	0.035	0.054	0.100	0.087
Minimum-Variance						
COV	0.095	0.132**	0.056	0.151**	0.182***	0.110*
EWMA	0.103	0.125	0.045	0.119	0.170***	0.094
VAR	0.093	0.134**	0.057	0.142**	0.191***	0.111*
VEC	(0.090*)	0.133**	0.056	0.146**	0.185***	0.113**
BEKK	0.098	0.132**	0.057	0.146**	0.181***	0.113**
ABEKK	0.098	0.131**	0.056	0.147**	0.181***	0.113**
CCC	0.100	0.131**	0.057	0.126**	0.176***	0.108**
DCC	0.100	0.132**	0.055	0.134**	0.172***	0.111**
ADCC	0.100	0.132**	0.055	0.134**	0.172***	0.111**
COPULA	0.100	0.129**	0.056	0.107*	0.118***	0.087
RSVAR	(0.087*)	0.101	0.041	0.084**	0.114***	0.098***
MSV	0.102	0.127	0.071	0.121*	0.158***	0.113
RCOV	0.108	0.137*	0.074	0.132**	0.198***	0.106
CP	0.089	0.107*	0.041	0.090**	0.117***	0.099***
Constrained Minimum-Variance						
COV	0.101	0.118*	0.044	0.099***	0.118***	0.099***
EWMA	0.112	0.121**	0.056	0.074	0.123***	0.094
VAR	0.099	0.119*	0.046	0.095**	0.120***	0.098**
VEC	0.099	0.119*	0.048	0.092**	0.117***	0.099***
BEKK	0.106	0.118*	0.046	0.098**	0.119***	0.098**
ABEKK	0.106	0.118*	0.046	0.098**	0.119***	0.098**
CCC	0.102	0.120**	0.047	0.098***	0.119***	0.099***
DCC	0.102	0.119*	0.043	0.098***	0.117***	0.098**
ADCC	0.102	0.119*	0.043	0.098**	0.117***	0.098***
COPULA	0.103	0.120**	0.044	0.089**	0.113***	0.090
RSVAR	0.087	0.102	0.041	0.084**	0.113***	0.098***
MSV	0.104	0.118*	0.060	0.086**	0.121***	0.099*
RCOV	0.111	0.123	0.054	0.100**	0.124***	0.098**
CP	0.090	0.107**	0.041	0.090**	0.115***	0.098***

Notes: See Table S3.1 for explanations of econometric model abbreviations; CP denotes combined parameter model (6) from the main paper. Dataset 1: Fama-French portfolios; Dataset 2: industry portfolios; Dataset 3: sector portfolios; Dataset 4: international equity indices; Dataset 5: portfolios sorted by size/book-to-market; Dataset 6: momentum portfolios. Results are for value-weighted data at weekly frequency. * significant at 10%; ** significant at 5%; *** significant at 1%. Significance corresponds to the Ledoit and Wolf (2008) robust test for differences between the Sharpe ratio and that of the naive strategy. Numbers in parentheses are statistically significantly worse than those of the naive strategy.

Table S1.2: Sharpe Ratio Net of Turnover Cost: Volatility-Timing

	Dataset 1	Dataset 2	Dataset 3	Dataset 4	Dataset 5	Dataset 6
Naïve	0.113	0.101	0.035	0.054	0.100	0.087
Volatility Timing						
COV	(0.088**)	0.109***	0.040	0.063**	0.105***	0.094***
EWMA	(0.085*)	0.111**	0.042	0.061	0.109***	0.093**
VAR	(0.081**)	0.109***	0.040	0.063**	0.106***	0.093***
VEC	(0.079**)	0.109***	0.040	0.062**	0.106***	0.094***
BEKK	(0.089*)	0.109***	0.040	0.063***	0.105***	0.093***
ABEKK	(0.089*)	0.109***	0.040	0.063***	0.105***	0.093***
CCC	(0.088**)	0.109***	0.040	0.063**	0.106***	0.094***
DCC	(0.088**)	0.109***	0.040	0.063***	0.105***	0.093***
ADCC	(0.088**)	0.109***	0.040	0.063**	0.105***	0.093***
COPULA	(0.088**)	0.109***	0.039	0.063***	0.105***	0.094***
RSVAR	(0.084*)	0.102	0.041	0.085**	0.113***	0.099***
MSV	(0.074**)	0.110**	0.044*	0.061*	0.108***	0.094**
RCOV	0.092	0.110***	0.043*	0.065***	0.105***	0.091***
CP	(0.085*)	0.103	0.040	0.076**	0.113***	0.098***

Notes: See Table S3.1 for explanations of econometric model abbreviations; CP denotes combined parameter model (6) from the main paper. Dataset 1: Fama-French portfolios; Dataset 2: industry portfolios; Dataset 3: sector portfolios; Dataset 4: international equity indices; Dataset 5: portfolios sorted by size/book-to-market; Dataset 6: momentum portfolios. Results are for value-weighted data at weekly frequency. * significant at 10%; ** significant at 5%; *** significant at 1%. Significance corresponds to the Ledoit and Wolf (2008) robust test for differences between the Sharpe ratio and that of the naive strategy. Numbers in parentheses are statistically significantly worse than those of the naive strategy.

S2 Pairwise Comparisons Between Variance Strategies

The minimum-variance strategy weakly dominates the volatility-timing strategy with higher Sharpe ratios and lower turnover costs for every econometric model and dataset. Other than dataset 1 (except for MSV), and RSVAR for dataset 2, the minimum-variance strategy weakly dominates the constrained minimum-variance strategy with higher Sharpe ratios, with and without controlling for turnover costs. Other than RCOV for dataset 1, and RSVAR and CP for datasets 1 and 2, the minimum-variance strategy weakly dominates the constrained minimum-variance strategy with lower portfolio volatility. Other than GARCHCOPULA for dataset 6, the constrained minimum-variance strategy weakly dominates the volatility-timing strategy with higher Sharpe ratios and lower turnover costs. Except RSVAR for dataset 2, both the minimum-variance strategy and the constrained minimum-variance strategy weakly dominate the volatility-timing strategy with lower portfolio volatility. Taken together, the minimum-variance strategy performs the best, with constrained minimum-variance in second place, and the volatility-timing strategy distinctly further down in third spot.

Table S2.1: Sharpe Ratio: Variance Strategies Pairwise Comparisons

	Dataset 1	Dataset 2	Dataset 3	Dataset 4	Dataset 5	Dataset 6
MVP vs Con-MVP						
Cov	$\times$	$\checkmark$*		$\checkmark$*	$\checkmark$	
EWMA	$\times$				$\checkmark$	
VAR	$\times$	$\checkmark$		$\checkmark$*	$\checkmark$	
VEC	$\times$				$\checkmark$	
BEKK	$\times$	$\checkmark$*		$\checkmark$*	$\checkmark$	$\checkmark$*
ABEKK	$\times$	$\checkmark$*		$\checkmark$*	$\checkmark$	$\checkmark$*
CCC	$\times$	$\checkmark$*			$\checkmark$	
DCC	$\times$*	$\checkmark$	$\checkmark$*		$\checkmark$	$\checkmark$*
ADCC	$\times$*	$\checkmark$*	$\checkmark$*		$\checkmark$	$\checkmark$*
GARCHCOPULA	$\times$	$\checkmark$	$\checkmark$*		$\checkmark$	
RSVAR	$\times$	$\times$	$\checkmark$*		$\checkmark$	$\checkmark$
MSV					$\checkmark$	
RCOV	$\times$		$\checkmark$*	$\checkmark$*	$\checkmark$	
CP	$\times$		$\checkmark$	$\checkmark$*	$\checkmark$	$\checkmark$
MVP vs VT						
Cov		$\checkmark$*		$\checkmark$	$\checkmark$	
EWMA	$\checkmark$*				$\checkmark$	
VAR	$\checkmark$	$\checkmark$*		$\checkmark$	$\checkmark$	$\checkmark$*
VEC	$\checkmark$	$\checkmark$*		$\checkmark$*	$\checkmark$	$\checkmark$*
BEKK		$\checkmark$*		$\checkmark$	$\checkmark$	$\checkmark$*
ABEKK		$\checkmark$*		$\checkmark$	$\checkmark$	$\checkmark$
CCC	$\checkmark$	$\checkmark$*		$\checkmark$	$\checkmark$	$\checkmark$*
DCC	$\checkmark$	$\checkmark$*		$\checkmark$	$\checkmark$	$\checkmark$*
ADCC	$\checkmark$	$\checkmark$		$\checkmark$	$\checkmark$	$\checkmark$*
GARCHCOPULA	$\checkmark$	$\checkmark$		$\checkmark$*	$\checkmark$	
RSVAR	$\checkmark$					
MSV	$\checkmark$				$\checkmark$	
RCOV	$\checkmark$			$\checkmark$	$\checkmark$	
CP	$\checkmark$	$\checkmark$		$\checkmark$	$\checkmark$	
con-MVP vs VT						
Cov	$\checkmark$			$\checkmark$	$\checkmark$	$\checkmark$
EWMA	$\checkmark$	$\checkmark$*			$\checkmark$	
VAR	$\checkmark$			$\checkmark$	$\checkmark$	$\checkmark$*
VEC	$\checkmark$			$\checkmark$	$\checkmark$	$\checkmark$
BEKK	$\checkmark$			$\checkmark$	$\checkmark$	$\checkmark$
ABEKK	$\checkmark$			$\checkmark$	$\checkmark$	$\checkmark$
CCC	$\checkmark$			$\checkmark$	$\checkmark$	$\checkmark$
DCC	$\checkmark$			$\checkmark$	$\checkmark$	$\checkmark$
ADCC	$\checkmark$			$\checkmark$	$\checkmark$	$\checkmark$
GARCHCOPULA	$\checkmark$	$\checkmark$*		$\checkmark$	$\checkmark$	$\times$
RSVAR	$\checkmark$					
MSV	$\checkmark$			$\checkmark$	$\checkmark$	
RCOV	$\checkmark$			$\checkmark$	$\checkmark$	
CP	$\checkmark$	$\checkmark$		$\checkmark$	$\checkmark$*	

Notes: See Table S3.1 for explanations of econometric model abbreviations; CP denotes combined parameter model (6) from the main paper; and MVP, con-MVP, and VT strategies are described in Table S4.1. See Table S1.1 for explanations of datasets. Results are for value-weighted data at weekly frequency. $\checkmark$: left strategy outperforms right at 5% level or lower; $\checkmark$*: left strategy outperforms right at 10% level; $\times$: left strategy underperforms right at 5% level or lower; $\times$*: left strategy underperforms right at 10% level; else, insignificant difference. Significance corresponds to the Ledoit and Wolf (2008) robust test for differences between the Sharpe ratios.

Table S2.2: Portfolio Volatility: Variance Strategies Pairwise Comparisons

	Dataset 1	Dataset 2	Dataset 3	Dataset 4	Dataset 5	Dataset 6
MVP vs con-MVP						
Cov					✓	✓
EWMA				✓*	✓	
VAR					✓	✓
VEC					✓	✓
BEKK					✓	✓
ABEKK					✓	✓
CCC					✓	✓
DCC					✓	✓
ADCC					✓	✓
GARCHCOPULA		✓*			✓*	✓
RSVAR	×	×	✓		✓	✓
MSV	✓*				✓	✓
RCOV	×*			✓*	✓	✓
CP	×	×	✓*		✓	✓
MVP vs VT						
Cov	✓	✓	✓	✓	✓	✓
EWMA	✓	✓		✓	✓	✓
VAR	✓	✓	✓	✓*	✓	✓
VEC	✓	✓	✓*	✓	✓	✓
BEKK		✓	✓*	✓*	✓	✓
ABEKK		✓	✓	✓*	✓	✓
CCC		✓	✓	✓	✓	✓
DCC	✓*	✓	✓	✓	✓	✓
ADCC	✓*	✓	✓	✓	✓	✓
GARCHCOPULA		✓	✓	✓*	✓	✓
RSVAR	✓	×				×
MSV	✓	✓		✓*	✓	✓
RCOV	✓	✓	✓*	✓	✓	✓
CP	✓	✓	✓	✓	✓	
con-MVP vs VT						
Cov	✓*	✓	✓	✓	✓	✓
EWMA	✓	✓	✓	✓	✓	✓
VAR	✓	✓	✓	✓	✓	✓
VEC	✓	✓	✓	✓	✓	✓
BEKK		✓	✓	✓	✓	✓
ABEKK		✓	✓	✓	✓	✓
CCC		✓	✓	✓	✓	✓
DCC		✓	✓	✓	✓	✓
ADCC		✓	✓	✓	✓	✓
GARCHCOPULA		✓	✓	✓	✓	✓
RSVAR	✓	×*				×
MSV	✓	✓	✓*	✓	✓	✓
RCOV	✓	✓	✓	✓	✓	✓
CP	✓	✓	✓	✓		

Notes: See Table S3.1 for explanations of econometric model abbreviations; CP denotes combined parameter model (6) from the main paper; and MVP, con-MVP, and VT strategies are described in Table S4.1. See Table S1.1 for explanations of datasets. Results are for value-weighted data at weekly frequency. ✓: left strategy outperforms right at 5% level or lower; ✓*: left strategy outperforms right at 10% level; ×: left strategy underperforms right at 5% level or lower; ×*: left strategy underperforms right at 10% level; else, insignificant difference. Significance corresponds to the Ledoit and Wolf (2011) robust test for unequal group variances.

Table S2.3: Sharpe Ratio Net of Turnover Cost: Variance Strategies Pairwise Comparisons

	Dataset 1	Dataset 2	Dataset 3	Dataset 4	Dataset 5	Dataset 6
MVP vs con-MVP						
Cov	$\times$	$\checkmark*$		$\checkmark*$	$\checkmark$	
EWMA	$\times$				$\checkmark$	
VAR	$\times$	$\checkmark$		$\checkmark*$	$\checkmark$	
VEC	$\times$				$\checkmark$	
BEKK	$\times$	$\checkmark*$		$\checkmark*$	$\checkmark$	$\checkmark*$
ABEKK	$\times$	$\checkmark*$		$\checkmark$	$\checkmark$	$\checkmark*$
CCC	$\times$				$\checkmark$	
DCC	$\times*$	$\checkmark*$			$\checkmark$	$\checkmark*$
ADCC	$\times*$	$\checkmark*$			$\checkmark$	$\checkmark*$
GARCHCOPULA	$\times$	$\checkmark$	$\checkmark*$		$\checkmark$	
RSVAR	$\times$	$\times$	$\checkmark*$		$\checkmark$	$\checkmark$
MSV					$\checkmark$	
RCOV	$\times$	$\checkmark*$	$\checkmark*$	$\checkmark*$	$\checkmark$	
CP	$\times$		$\checkmark$	$\checkmark*$	$\checkmark$	$\checkmark$
MVP vs VT						
Cov		$\checkmark*$		$\checkmark$	$\checkmark$	$\checkmark*$
EWMA	$\checkmark$				$\checkmark$	
VAR	$\checkmark*$	$\checkmark$		$\checkmark$	$\checkmark$	
VEC	$\checkmark$	$\checkmark*$		$\checkmark$	$\checkmark$	$\checkmark*$
BEKK	$\checkmark*$	$\checkmark*$		$\checkmark$	$\checkmark$	$\checkmark*$
ABEKK		$\checkmark*$		$\checkmark$	$\checkmark$	$\checkmark$
CCC	$\checkmark$	$\checkmark*$		$\checkmark$	$\checkmark$	$\checkmark*$
DCC	$\checkmark$	$\checkmark*$		$\checkmark$	$\checkmark$	$\checkmark*$
ADCC	$\checkmark$	$\checkmark*$		$\checkmark$	$\checkmark$	$\checkmark*$
GARCHCOPULA	$\checkmark$	$\checkmark$		$\checkmark*$	$\checkmark$	
RSVAR	$\checkmark$					
MSV	$\checkmark$			$\checkmark*$	$\checkmark$	
RCOV	$\checkmark$			$\checkmark$	$\checkmark$	
CP	$\checkmark$	$\checkmark$		$\checkmark$	$\checkmark$	
con-MVP vs VT						
Cov	$\checkmark$			$\checkmark$	$\checkmark$	$\checkmark$
EWMA	$\checkmark$	$\checkmark*$			$\checkmark$	
VAR	$\checkmark$			$\checkmark$	$\checkmark$	$\checkmark$
VEC	$\checkmark$			$\checkmark$	$\checkmark$	$\checkmark$
BEKK	$\checkmark$			$\checkmark$	$\checkmark$	$\checkmark$
ABEKK	$\checkmark$			$\checkmark$	$\checkmark$	$\checkmark$
CCC	$\checkmark$			$\checkmark$	$\checkmark$	$\checkmark$
DCC	$\checkmark$			$\checkmark$	$\checkmark$	$\checkmark$
ADCC	$\checkmark$			$\checkmark$	$\checkmark$	$\checkmark*$
GARCHCOPULA	$\checkmark$	$\checkmark*$		$\checkmark$	$\checkmark$	$\times$
RSVAR	$\checkmark$					
MSV	$\checkmark$			$\checkmark$	$\checkmark$	
RCOV	$\checkmark$			$\checkmark$	$\checkmark$	
CP	$\checkmark$	$\checkmark$		$\checkmark$	$\checkmark*$	

Notes: See Table S3.1 for explanations of econometric model abbreviations; CP denotes combined parameter model (6) from the main paper; and MVP, con-MVP, and VT strategies are described in Table S4.1. See Table S1.1 for explanations of datasets. Results are for value-weighted data at weekly frequency. $\checkmark$: left strategy outperforms right at 5% level or lower; $\checkmark*$: left strategy outperforms right at 10% level; $\times$: left strategy underperforms right at 5% level or lower; $\times*$: left strategy underperforms right at 10% level; else, insignificant difference. Significance corresponds to the Ledoit and Wolf (2008) robust test for differences between the Sharpe ratios.

S3 Econometric Models

Table S3.1 lists the econometric models.

S3.1 Sample Covariance

Our first econometric model to estimate the variance-covariance matrix is the sample covariance matrix (Cov). Sample-based in-sample estimation of the variance-covariance matrix is standard in the literature on the naive diversification puzzle (DeMiguel *et al.*, 2009; Fletcher, 2011; Tu and Zhou, 2011; Kirby and Ostdiek, 2012). Some studies examine improved estimation, but they typically focus on a small set of models (DeMiguel *et al.*, 2013). The sample covariance matrix places equal weight on past observations.

S3.2 Exponentially Weighted Moving Average

The recent past might be more informative for estimating the variance-covariance matrix to use in selecting portfolios, motivating our first refinement, the exponentially weighted moving average (EWMA) model. The EWMA model suggested by RiskMetrics places decaying weight on the past: $\Sigma_t = \alpha\Sigma_{t-1} + (1-\alpha)(\mathbf{r}_t - \bar{\mathbf{r}}_t)'(\mathbf{r}_t - \bar{\mathbf{r}}_t)$ where our decay parameter suggested is 0.96 (weekly). The EWMA specification corresponds to a nonstationary IGARCH model with zero intercept.

S3.3 Vector Autoregression

To exploit dependence along the cross-section and time-dimension (serial), we estimate a vector autoregression (VAR): $Y = X\Phi + U$ using Bayesian methods. We experimented with a VAR with a Normal-Wishart natural conjugate prior and another with a Minnesota prior. To reduce the impact of our prior on our results, we choose the posterior based on a non-informative prior, and we report the posterior mode of the variance-covariance matrix. Long lags are helpful to approximate the Wold representation, but we typically find two lags to be optimal at weekly frequency, especially as we look at financial variables and given computational considerations. We therefore include a constant and two lags of the variables, i.e., we estimate a VAR(2).

S3.4 Vector Error Correction

Accounting for potential cointegration between the variables, we move beyond VAR to estimate a parsimonious vector error correction (VEC) model: $\Delta\mathbf{y}_t = \Pi\mathbf{y}_{t-1} + \sum_{i=1}^{p-1}\Phi_i^*\Delta\mathbf{y}_{t-i} + \boldsymbol{\epsilon}_t$. We compute the number of cointegrating relations in the system following the Johansen trace test (Johansen, 1988, 1991) and employ the variance-covariance matrix estimated from the VEC model. If the Johansen test fails to reject the null of no cointegrating relations, then our VEC reduces to a VAR in first differences (error-correction coefficient is zero).

S3.5 BEKK-GARCH and Asymmetric BEKK-GARCH

The volatility of financial return data varies over time. General Autoregressive Conditional Heteroscedasticity (GARCH) provides a simple way of modeling the evolution of volatility as a deterministic function of past volatility and innovations. Univariate GARCH expresses the error term of a time series $\epsilon_t = H_t^{1/2}\xi_t$ where ξ_t is an i.i.d. innovation and

$H_t = f(\{\epsilon_{t-i}, H_{t-j}\}_{i=1,j=1}^{q,p})$ and can be extended to the multivariate setting through the $vech(\cdot)$ operator that stacks the lower triangular part of a symmetric $N \times N$ matrix into a $N^* = N(N+1)/2$ dimensional vector: $vech(H_t) = \boldsymbol{\omega} + \sum_{i=1}^{q} A_i \boldsymbol{\eta}_{t-i} + \sum_{j=1}^{p} B_j \mathbf{h}_{t-j}$. With this notation, $\mathbf{h}_t = vech(H_t)$, $\boldsymbol{\omega}$ is the constant component of the covariances, and $\boldsymbol{\eta}_t = vech(\boldsymbol{\epsilon}_t \boldsymbol{\epsilon}_t^T)$. To prepare the data for GARCH estimation, we fit an ARMA model to the data for each series from which to obtain demeaned residuals ϵ_t. Diagnostics such as Ljung Box tests of serial correlation suggest ARMA(1,1) fits the data well.

Our first multivariate GARCH specification is BEKK-GARCH (Engle and Kroner, 1995), which allows for the dependence of conditional variances of one variable on lagged values of another so that causalities in variances can be modeled. Empirically, BEKK is general, but easy to estimate. Relaxing symmetry, we also allow positive and negative shocks of equal magnitude to have different effects on conditional volatility by employing Asymmetric BEKK (ABEKK). With ABEKK, $vech(H_t) = \boldsymbol{\omega} + \sum_{i=1}^{q} [A_i \boldsymbol{\epsilon}_{t-i} + C_i \boldsymbol{\eta}_{t-i}] + \sum_{j=1}^{p} B_j \mathbf{h}_{t-j}$. We allow ($q = 1$) one symmetric innovation when estimating BEKK. When estimating ABEKK, we allow one symmetric innovation and one asymmetric innovation.

S3.6 Conditional Correlation: Constant, Dynamic, & Asymmetric

BEKK and its sibling ABEKK suffer from the curse of dimensionality that might render them computationally infeasible for investors allocating capital across a large set of investment choices. We therefore also estimate a constant conditional correlation (CCC) model. The CCC model is a multivariate GARCH model, where all conditional correlations are constant and conditional variances are modeled by univariate GARCH processes (Bollerslev *et al.*, 1990).

Let us decompose the conditional covariance matrix into conditional standard deviations and a correlation matrix $H_t = D_r R_t D_t$ where $D_t = \text{diag}\left(h_{1t}^{1/2}, \ldots, h_{Nt}^{1/2}\right)$ is a diagonal matrix of the standard deviations for the N investments. The conditional correlation matrix for the CCC model is constant over time: $R_t = R$. The CCC model benefits from almost unrestricted applicability for large systems of time series, but fails to account for the observation that correlation increases during financial crises.

Dynamic conditional correlation (DCC) permits correlation to vary over time (Engle, 2002). In addition to DCC, we also estimate its asymmetric version (ADCC). Without accounting for dynamics of asymmetric effects, DCC cannot distinguish between the effect of past positive and negative shocks on the future conditional volatility and levels (Cappiello *et al.*, 2006). As when we estimate BEKK and ABEKK, we allow ($q = 1$) one symmetric innovation when estimating CCC and DCC. When estimating ADCC, we allow one symmetric innovation and one asymmetric innovation.

S3.7 Copula-GARCH

The assumption of multivariate normality is often called into question in practical applications. As motivation, consider Apple and Microsoft, who produce similar products. Shocks that affect Apple may be expected to affect Microsoft. Each company may experience similar nonlinear extreme events, hence exhibiting tail dependence. A portfolio manager who assumes multivariate normality will underestimate the frequency and magnitude of rare events. Such underestimation may be detrimental to the performance of the portfolio.

Modeling multivariate dependence among stock returns without assuming multivariate normality has become popular in the 21st century. Copulas are functions that may be used to bind univariate marginal distributions to produce a multivariate distribution (Sklar, 1959). Parameters can vary over time as an autoregression in a copula-GARCH model. Copulas have become the standard tools for modeling multivariate dependence among stock returns without assuming multivariate normality with many general applications in finance (Stric and Granger, 2005; Zimmer, 2012; Christoffersen *et al.*, 2012; Aloui *et al.*, 2013; Christoffersen and Langlois, 2013; Creal *et al.*, 2013; Xiao, 2014; Adrian and Brunnermeier, 2016; Bodnar and Hautsch, 2016; Solnik and Watewai, 2016; Bekiros *et al.*, 2017).

We specify a copula-GARCH process without fitting a VAR for the conditional mean. We set the following tuning parameters to the robust regression: $\gamma = 0.25$ (proportion to trim), $\delta = 0.01$ (critical value for the re-weighted estimator), 500 subsets, and 10 steps. We allow for a symmetric DCC autoregressive order of (1,1) and choose the multivariate Student copula distribution model, where the DCC copula is static, and we estimate the correlation parameter in the static Student copula by maximum likelihood. We apply an empirical (pseudo maximum likelihood) transformation to the marginal innovations of the GARCH fitted models. In estimating the above specification for return data, we calculate standard errors, require stationarity when optimizing the univariate GARCH, and do not use any scale option during this first stage.[1] We take the average robust estimate of the covariance matrix.

S3.8 Regime-Switching Vector Autoregression

To account for bull and bear phases of the market, we estimate a discrete time-varying parameter model in the form of a regime-switching VAR (RSVAR) as in Chan and Eisenstat (2018): $B_{0,S_t}\mathbf{y}_t = \boldsymbol{\mu}_{S_t} + B_{1,S_t}\mathbf{y}_{t-1} + \cdots + B_{p,S_t}\mathbf{y}_{t-p} + \boldsymbol{\epsilon}_t$ where $\boldsymbol{\epsilon}_t \sim \mathcal{N}(\mathbf{0}, \Sigma_{S_t})$. We choose two regimes in the mean and in the covariance matrix. For each rolling window, we use a pre-sample of one year and estimate over the remaining sample of nine years in the window. Our Bayesian estimation includes 20,000 simulations with a burn-in of 5,000 periods. We set our lag length at 2 for parsimony. To derive the variance-covariance matrix from the RSVAR, we back out the states using highest probability, get part of the parameter set Θ corresponding to the state at each time, and calculate the variance-covariance matrix as $\Sigma = (Y - X\Theta)'(Y - X\Theta)$.

S3.9 Multivariate Stochastic Volatility

Another nonlinear state-space model that allows for heteroscedasticity is the computationally challenging multivariate stochastic volatility model (MSV). The curse of dimensionality for the MSV is that the degrees of freedom in the variance-covariance matrix scales quadratically with the number of investments. Multivariate factor stochastic volatility breaks the curse of dimensionality by decomposing the variance-covariance matrix and using the pivoted Cholesky algorithm of Higham (1990). The decomposition transforms the estimation problem from

[1] We alternate between using R's *solnp* solver, which is a nonlinear optimization using the augmented Lagrange method, and *gosolnp*, which randomly initializes and conducts multiple restarts of the *solnp* solver. When the objective function is non-smooth or has many local minima, it is hard to judge the optimality of the solution, and this usually depends critically on the starting parameters. The *gosolnp* function enables the generation of a set of randomly chosen parameters from which to initialize multiple restarts of the solver. We chose *solnp*, as it is faster, but when our solver encounters difficulties, we switch to *gosolnp*.

being quadratic in investments to becoming linear in investments.[2]

We demean the data and use $r = 2$ factors, where the factor loadings are unidentified. As the sampler places no constraints on the loading matrix, we benefit from significant reductions in run time.[3] Placing no constraints on the loading matrix possibly results in unstable posteriors or multiple local optima. The instability of posterior estimates or multiplicity of local optima cause no issues for our study, however, as inference is on the covariance matrix rather than on the factor loadings (Kastner, 2019a). We find 10,000 draws with a burn in of 1,000 sufficient for convergence of the estimates.

S3.10 Realized Volatility

Our final econometric approach is distribution or model free: realized volatility (RCOV). Realized variance is the summation matrix of the return vector outerproduct $\mathbf{r}_i'\mathbf{r}_i$ over each day in a given week. Realized volatility is the Cholesky decomposition of the realized variance. We focus on annualized realized volatility by pre-multiplying the realized variance by the number of trading days in a year divided by the number of trading days in a week and obtaining the Cholesky decomposition of this product.

[2]We adapt our method and exposition from Kastner (2019a,b). Observations of returns $\mathbf{y}_t = (y_{t1}, \ldots, y_{tN})'$ follow

$$\begin{aligned} \mathbf{y}_t|\Lambda, \mathbf{f}_t, \bar{\Sigma}_t &\sim \mathcal{N}_N(\Lambda\mathbf{f}_t, \bar{\Sigma}_t) \\ \mathbf{f}_t|\tilde{\Sigma}_t &\sim \mathcal{N}(\mathbf{0}, \tilde{\Sigma}_t), \end{aligned}$$

where $\mathbf{f}_t = (f_{t1}, \ldots, f_{tr})'$ is the vector of factors and $\Lambda \in \mathbb{R}^{N\times r}$ is the matrix of factor loadings. The covariance matrices $\bar{\Sigma}_t$ and $\tilde{\Sigma}_t$ are diagonal and represent stochastic volatility processes

$$\begin{aligned} \bar{\Sigma}_t &= \operatorname{diag}(\exp \bar{h}_{t1}, \ldots, \exp \bar{h}_{tN}) \\ \tilde{\Sigma}_t &= \operatorname{diag}(\exp \tilde{h}_{t1}, \ldots, \exp \tilde{h}_{tr}) \\ \bar{h}_{ti} &\sim \mathcal{N}(\bar{\mu}_i + \bar{\psi}_i(\bar{h}_{t-1,i} - \bar{\mu}_i), \bar{\sigma}_i^2) \quad i = 1, \ldots, N \\ \tilde{h}_{tj} &\sim \mathcal{N}(\tilde{\mu}_j + \tilde{\psi}_j(\tilde{h}_{t-1,j} - \tilde{\mu}_j), \tilde{\sigma}_j^2) \quad j = 1, \ldots, r. \end{aligned}$$

With latent factor models, few shocks drive the system and we can reduce the number of unknowns through the decomposition $\Sigma_t = \tilde{\Sigma}_t + \bar{\Sigma}_t$ where $\operatorname{rank}(\tilde{\Sigma}) = r < N$ and $\bar{\Sigma}_t$ is a diagonal matrix where the diagonal entries are the idiosyncratic errors. Using the pivoted Cholesky algorithm of Higham (1990), $\tilde{\Sigma}_t = \Psi\Psi'$ where $\Psi \in \mathbb{R}^{N\times r}$ has $Nr - r(r-1)/2$ free elements; therefore, $N(r+1) - r(r-1)/2$ free elements are left in Σ_t, which is now linear in N. Thus, $\Sigma_t = \Lambda\tilde{\Sigma}_t\Lambda_t + \bar{\Sigma}_t$.

[3]We also speed up computation by storing only the conditional covariance matrix and the square roots of its diagonal elements and parallelizing the *factorstochvol* function of Kastner (2019a) in R and C++ with hyperthreading.

Table S3.1: Econometric models

1	Sample covariance matrix (Cov)
2	Exponentially weighted covariance matrix (EWMA)
3	Vector autoregressions (VAR)
4	Vector error corrections (VEC)
5	Multivariate BEKK-GARCH (BEKK)
6	Asymmetric multivariate BEKK-GARCH (ABEKK)
7	Constant conditional correlation GARCH (CCC)
8	Dynamic conditional correlation GARCH (DCC)
9	Asymmetric dynamic conditional correlation GARCH (ADCC)
10	T-copula with GARCH margins (COPULA)
11	Regime-switching vector autoregressions (RSVAR)
12	High dimensional multivariate stochastic volatility (MSV)
13	Realized covariance (RCOV)

S4 Portfolio Models

Table S4.1: Portfolio models

1	Naive portfolio model based on equal investment across investments (1/N)
2	Minimum-variance portfolio (MVP)
3	Constrained minimum-variance portfolio (Con-MVP)
4	Volatility-timing strategies (VT)
5	Naive portfolio inputting average cov-matrix in Table S3.1 into each of M2-M5 (CP)
6	Naive portfolio across each of the cov-matrices in Table S3.1 based on each of M2-M5
7	Naive portfolio across M2-M5 based on each of the cov-matrices in Table S3.1

Notes: We describe combined parameter (CP) model 5 in equation (6) and the other combined parameter models 6 and 7 in footnote (16) of the main paper.

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