Problem Set 2: Volatility & Filtering

Volatility

ARCH, GARCH and Forecasting

Exercise 1 (50 Marks)**.** For this exercise you will need the dataset tsdata2.mat and the problems MUST be implemented in Matlab where indicated. For this you will need to provide your Matlab program in a separate sheet and please highlight the changes you did to the original program. Since the following exercises should be implemented for two different portfolios, you only need to provide the Matlab code for one portfolio. Let the stock market index of country i be denoted as P_{it} and let R_{it} denote the EUR return of a portfolio of British, German and US stocks. Construct the log return series for each country as $\triangle R_{it} = lnP_{it} - lnP_{it-1}$. Construct a EUR return portfolio, R_{pt} with equal country weights (don't forget to account for the exchange rate changes).

a) For model comparisons, construct a baseline measure of daily volatility:

$$\sigma_t = \sqrt{y_t^2} \tag{1}$$

where y_t is i) the portfolio return, R_{pt} ii) the Germany total return index P_{1t}. (1 point)

b) Consider the following Exponential Smoothing model of the variance:

$$\sigma_{t+1}^2 = (1-\lambda)y_t^2 + \lambda\sigma_t^2 \tag{2}$$

Estimate λ using MLE. Construct an estimated series of period t's variance, $\hat{\sigma}_{t+1}^2 = (1-\lambda)y_t^2 + \lambda\sigma_t^2$ and of period t's volatility. Plot the series against the baseline. Why is the correlation between the two series so high? Re-estimate λ based on the first half of the sample. Construct an estimate, $\hat{\sigma}_{t+1}^2$ using the second half of the sample. Plot the series against the baseline. Why is the correlation between the two series so high? Based on the estimate of λ, can we predict changes in tomorrow's volatility? (4 points)

c) Consider the following GARCH(2,1)model of the variance:

$$\sigma_{t+1}^2 = \omega + \alpha_1 y_t^2 + \alpha_2 y_{t-1}^2 + \beta\sigma_t^2 \tag{3}$$

Estimate the parameters ω, α_1, α_2 and β using MLE. Construct an estimated series of period t's variance, $\hat{\sigma}_{t+1}^2 = \omega + \alpha_1 y_t^2 + \alpha_2 y_{t-1}^2 + \beta\sigma_t^2$ and of period t's volatility. Plot the series against the baseline. Why is the correlation between the two series so high? Re-estimate the parameters based on the first half of the sample. Construct an estimate, $\hat{\sigma}_{t+1}^2$ using the second half of the sample. Plot the series against the baseline. Why is the correlation between the two series so high? Based on the estimated parameters, can we predict changes in tomorrow's volatility? (20 points)

d) Consider the following leverage GARCH(1,2) model of the variance:

$$\sigma_{t+1}^2 = \omega + \alpha_1(y_t - \alpha_2\sigma_t)^2 + \beta_1\sigma_t^2 + \beta_2\sigma_{t-1}^2 \tag{4}$$

Estimate the parameters ω, α_1, α_2, β_1 and β_2 using MLE. Construct an estimated series of period t's variance, $\hat{\sigma}_{t+1}^2 = \omega + \alpha_1(y_t - \alpha_2\sigma_t)^2 + \beta_1\sigma_t^2 + \beta_2\sigma_{t-1}^2$ and of period t's volatility. Plot the series against the baseline. Why is the correlation between the two series so high? Re-estimate the parameters based on the first half of the sample. Construct an estimate, $\hat{\sigma}_{t+1}^2$ using the second half of the sample. Plot the series against the baseline. Why is the correlation between the two series so high? Based on the estimated parameters, can we predict changes in tomorrow's volatility? (20 points)

e) Compare the above models. How do they perform against their GARCH (1,1) counterpart (5 points)?

GARCH

Exercise 2 (12 Marks)**.** Consider the GARCH model

$$\begin{aligned} y_t &= \sigma_t \epsilon_t \quad \epsilon_t \sim NID(0,1) \\ \sigma_t^2 &= \gamma + \alpha y_{t-1}^2 + \beta \sigma_{t-1}^2 \quad \gamma > 0, \alpha \geq 0, \beta \geq 0 \end{aligned}$$

1. Show that y_t is a martingale difference and derive an expression for its (unconditional) variance. (4 Marks)
2. Find the autocorrelation function of (a) y_t and (b) y_t^2. Discuss. (6 Marks)
3. Write down the log-likelihood function for the GARCH-M model

$$y_t = \delta \sigma_t + u_t \quad t = 1, \ldots, T$$

in which δ is an unknown parameter and u_t follows a GARCH(1,1) process. State any assumptions you make. (2 Marks)

Markov-Switching & Stochastic Volatility Models

Exercise 3 (Optional)**.**

1. Why might we want to consider employing Markov-Switching models? If one was to entertain the possibility of using such models, briefly describe the basic setup, how optimal inference and forecasting might be conducted in a recursive manner, how we might start the algorithm and how we might estimate the parameters.
2. Why might we want to consider employing Stochastic-Volatility models? If one was to entertain the possibility of using such models, briefly describe the basic setup, a few properties of the model and some issues with the model.

Filtering

State Space Form & Kalman Filtering

Exercise 4 (20 Marks)**.** Consider the following MA(2) model

$$y_t = \epsilon_t + \theta_1 \epsilon_{t-1} + \epsilon_{t-2}$$

Write this model in state space form. (4 Marks) Is the model stationary? (1 Mark) How would you initialise the Kalman filter? (1 Mark) Write down the initial state vector $\mathbf{a}_0$, the initial MSE matrix $\mathbf{P}_0$, the first prediction error v_1 and its MSE f_1. (4 Marks) What would the updated state vector $\mathbf{a}_1$ and covariance matrix $\mathbf{P}_1$ be? (2 Marks) Write the prediction equations $\mathbf{a}_{2|1}$ and $\mathbf{P}_{2|1}$ and v_2 and f_2. (8 Marks)
Optional: derive the recursion for the Kalman filter in this case for v_t and f_t.

ACGF & Spectrums

Exercise 5 (4 Marks)**.** Derive the autocovariance generating function and the spectral density function for the MA(1) process

$$y_t = \epsilon_t + \theta\epsilon_{t-1}$$

Hint: the MA(1) process may be expressed as $\Phi(L)y_t = \Theta(L)\epsilon_t$ where the lag polynomials are $\Theta(e^{i\omega}) = 1+e^{i\omega}$ and $\Phi(e^{i\omega}) = 1$.

Seasonal difference filter

Exercise 6 (2 Marks)**.** Derive the gain function for the seasonal difference filter, $h(L) = 1 - L^{12}$.

Real time filtering & filtered data

Exercise 7 (5 Marks)**.**

1. Suppose you are asked to estimate the output gap in real time. How might you go about this and what issues might be present in using two-sided filters? (3 Marks)
2. Discuss some issues in conducting regressions using filtered data. (2 Marks)

Hodrick-Prescott filter

Exercise 8 (7 Marks)**.** The gain function of the Hodrick-Prescott filter is given by

$$G(\omega) = \left[1 + \left(\frac{\sin(\omega/2)}{\sin(\omega_0/2)}\right)^4\right]^{-1}$$

where

$$\omega_0 = 2arcsin\left(\frac{1}{2\lambda^{\frac{1}{4}}}\right)$$

Different specifications of λ imply that 50% of the filter gain has been completed at a particular number of cycles, e.g. 40-quarter cycle.

Using MATLAB, determine the values of λ for which 50% of the filter gain has been completed at

1. 40-quarter cycle (10 years for quarterly data) (1 Mark)
2. 10-year cycle (annual data) (1 Mark)
3. 120-month cycle (monthly data) (1 Mark)
4. 32-quarter cycle (1 Mark)
5. 6-quarter cycle (1 Mark)
6. 20-quarter cycle (1 Mark)
7. 56-quarter cycle (1 Mark)