Problem Set 2: Volatility & Filtering

Volatility

ARCH, GARCH and Forecasting

Exercise 1 (50 Marks)**.** For this exercise you will need the dataset tsdata2.mat and the problems MUST be implemented in Matlab where indicated. For this you will need to provide your Matlab program in a separate sheet and please highlight the changes you did to the original program. Since the following exercises should be implemented for two different portfolios, you only need to provide the Matlab code for one portfolio. Let the stock market index of country i be denoted as P_{it} and let R_{it} denote the EUR return of a portfolio of British, German and US stocks. Construct the log return series for each country as $\triangle R_{it} = lnP_{it} - lnP_{it-1}$. Construct a EUR return portfolio, R_{pt} with equal country weights (don't forget to account for the exchange rate changes).

a) For model comparisons, construct a baseline measure of daily volatility:

$$\sigma_t = \sqrt{y_t^2} \tag{1}$$

where y_t is i) the portfolio return, R_{pt} ii) the Germany total return index P_{1t}. (1 point)

b) Consider the following Exponential Smoothing model of the variance:

$$\sigma_{t+1}^2 = (1-\lambda)y_t^2 + \lambda\sigma_t^2 \tag{2}$$

Estimate λ using MLE. Construct an estimated series of period t's variance, $\hat{\sigma}_{t+1}^2 = (1-\lambda)y_t^2 + \lambda\sigma_t^2$ and of period t's volatility. Plot the series against the baseline. Why is the correlation between the two series so high? Re-estimate λ based on the first half of the sample. Construct an estimate, $\hat{\sigma}_{t+1}^2$ using the second half of the sample. Plot the series against the baseline. Why is the correlation between the two series so high? Based on the estimate of λ, can we predict changes in tomorrow's volatility? (4 points)

c) Consider the following GARCH(2,1)model of the variance:

$$\sigma_{t+1}^2 = \omega + \alpha_1 y_t^2 + \alpha_2 y_{t-1}^2 + \beta\sigma_t^2 \tag{3}$$

Estimate the parameters ω, α_1, α_2 and β using MLE. Construct an estimated series of period t's variance, $\hat{\sigma}_{t+1}^2 = \omega + \alpha_1 y_t^2 + \alpha_2 y_{t-1}^2 + \beta\sigma_t^2$ and of period t's volatility. Plot the series against the baseline. Why is the correlation between the two series so high? Re-estimate the parameters based on the first half of the sample. Construct an estimate, $\hat{\sigma}_{t+1}^2$ using the second half of the sample. Plot the series against the baseline. Why is the correlation between the two series so high? Based on the estimated parameters, can we predict changes in tomorrow's volatility? (20 points)

d) Consider the following leverage GARCH(1,2) model of the variance:

$$\sigma_{t+1}^2 = \omega + \alpha_1(y_t - \alpha_2\sigma_t)^2 + \beta_1\sigma_t^2 + \beta_2\sigma_{t-1}^2 \tag{4}$$

Estimate the parameters ω, α_1, α_2, β_1 and β_2 using MLE. Construct an estimated series of period t's variance, $\hat{\sigma}_{t+1}^2 = \omega + \alpha_1(y_t - \alpha_2\sigma_t)^2 + \beta_1\sigma_t^2 + \beta_2\sigma_{t-1}^2$ and of period t's volatility. Plot the series against the baseline. Why is the correlation between the two series so high? Re-estimate the parameters based on the first half of the sample. Construct an estimate, $\hat{\sigma}_{t+1}^2$ using the second half of the sample. Plot the series against the baseline. Why is the correlation between the two series so high? Based on the estimated parameters, can we predict changes in tomorrow's volatility? (20 points)

e) Compare the above models. How do they perform against their GARCH (1,1) counterpart (5 points)?

Solution 1 (ARCH, GARCH & Forecasting)**.**
See MATLABps2sol.pdf.

GARCH

Exercise 2 (12 Marks)**.** Consider the GARCH model

$$\begin{aligned} y_t &= \sigma_t \epsilon_t \quad \epsilon_t \sim NID(0,1) \\ \sigma_t^2 &= \gamma + \alpha y_{t-1}^2 + \beta \sigma_{t-1}^2 \quad \gamma > 0, \alpha \geq 0, \beta \geq 0 \end{aligned}$$

1. Show that y_t is a martingale difference and derive an expression for its (unconditional) variance. (4 Marks)
2. Find the autocorrelation function of (a) y_t and (b) y_t^2. Discuss. (6 Marks)
3. Write down the log-likelihood function for the GARCH-M model

$$y_t = \delta \sigma_t + u_t \quad t = 1, \ldots, T$$

   in which δ is an unknown parameter and u_t follows a GARCH(1,1) process. State any assumptions you make. (2 Marks)

Solution 2 (GARCH models)**.**
Consider the following model for parts (i) and (ii)

$$\begin{aligned} y_t &= \sigma_t \epsilon_t \quad \epsilon_t \sim NID(0,1) \\ \sigma_t^2 &= \gamma + \alpha y_{t-1}^2 + \beta \sigma_{t-1}^2 \quad \gamma > 0, \alpha \geq 0, \beta \geq 0 \end{aligned} \tag{5}$$

Part (i) Moments. A process y_t is a martingale difference if $E_{t-1}[y_t] = 0$ where E_{t-j} is the expectation with information up to and including period $t-j$. In the case of the GARCH model (5), since σ_t^2 is built with information from period $t-1$, then $E_{t-1}[\sigma_t] = \sigma_t$. Hence,

$$E_{t-1}[y_t] = E_{t-1}[\sigma_t] E_{t-1}[\epsilon_t] = \sigma_t \cdot 0 = 0 \tag{6}$$

so y_t is indeed a martingale difference.

Recall the law of iterated expectations for computing the variance of y_t. By definition the unconditional expectation of some quantity x is

$$E[x] = \lim_{j \longrightarrow \infty} E_{t-j} \ldots E_{t-3} E_{t-2} E_{t-1}[x] \tag{7}$$

From (6) & (7), it is clear that $E[y_t] = 0$ so the variance of y_t will be $E[y_t^2]$. Note also that $E_{t-j}[y_t^2] = E_{t-j}[\sigma_t^2]$. Then

$$\begin{aligned} E_{t-1}[y_t^2] &= \gamma + \alpha y_{t-1}^2 + \beta \sigma_{t-1}^2 \\ E_{t-2}E_{t-1}[y_t^2] &= \gamma + (\alpha+\beta)(\gamma + \alpha y_{t-2}^2 + \beta \sigma_{t-2}^2) \\ E_{t-3}E_{t-2}E_{t-1}[y_t^2] &= \gamma(1+\alpha+\beta) + (\alpha+\beta)^2(\gamma + \alpha y_{t-3}^2 + \beta \sigma_{t-3}^2) \\ E_{t-4}E_{t-3}E_{t-2}E_{t-1}[y_t^2] &= \gamma(1+(\alpha+\beta)+(\alpha+\beta)^2) + (\alpha+\beta)^3(\gamma + \alpha y_{t-4}^2 + \beta \sigma_{t-4}^2) \end{aligned}$$

so the general pattern of iterated expectations for some $j > 2$ is

$$E_{t-j} \dots E_{t-2}E_{t-1}[y_t^2] = \gamma(1 + (\alpha+\beta) + \cdots + (\alpha+\beta)^{j-2}) + (\alpha+\beta)^{j-1}\sigma_{t-j+1}^2$$

Therefore, making the assumption that $(\alpha+\beta) < 1$, the variance of y_t is

$$E[y_t^2] = \lim_{j \longrightarrow \infty} E_{t-j} \dots E_{t-2}E_{t-1}[y_t^2] = \frac{\gamma}{1-\alpha-\beta}$$

Part (ii) Autocorrelations. The levels of y_t appear to be a white noise (no autocorrelation at any lag). To see this, note that for $\tau > 0$

$$E[y_t y_{t-\tau}] = E[\epsilon_t \epsilon_{t-\tau}\sigma_t\sigma_{t-\tau}] = E[\epsilon_t]E[\epsilon_{t-\tau}\sigma_t\sigma_{t-\tau}] = 0$$

However, y_t^2 follows a certain ARMA process and thus past information can be used to forecast it. To find the autocorrelations of y_t^2 we follow Harvey (1993:276) and define the variable $v_t = \sigma_t^2(\epsilon_t^2 - 1)$ that is a white noise since

$$E_{t-1}(\sigma_t^2(\epsilon_t^2 - 1)) = \sigma_t^2[E_{t-1}(\epsilon_t^2) - 1] = 0$$

Hence

$$y_t^2 = \sigma_t^2 + v_t \tag{8}$$

From (5) we have that $(1-\beta L)\sigma_t^2 = \gamma + \alpha y_{t-1}^2$. Then, after multiplying (8) by $(1-\beta L)$,

$$y_t^2 = \gamma + (\alpha+\beta)y_{t-1}^2 + v_t - \beta v_{t-1}$$

so y_t^2 follows an ARMA(1,1) process. From our knowledge on ARMA models we then get that

$$\rho_1 = \frac{\alpha(1-(\alpha+\beta)\beta)}{(1-2(\alpha+\beta)\beta+\beta^2)}$$

and

$$\rho_\tau = (\alpha+\beta)\rho_{\tau-1} \quad \text{for } \tau \geq 2$$

To see that this is the autocorrelation function for an ARMA(1,1) process, consider the general ARMA(1,1) process

$$y_t = \phi y_{t-1} + \epsilon_t + \theta\epsilon_{t-1} \tag{9}$$

where we assume $|\phi| < 1$ to ensure stationarity. The following properties result

$$E[y_t\epsilon_s] = \begin{cases} \sigma^2 & \text{if } s = t \\ (\phi+\theta)\sigma^2 & \text{if } s = t-1 \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

After multiplying (9) by y_t, taking expectations and using (10), we get that

$$\gamma_0 = \phi\gamma_1 + \sigma^2 + \theta(\phi+\theta)\sigma^2 \tag{11}$$

Now multiply (9) by y_{t-1}, take expectations and use (10) to get

$$\gamma_1 = \phi\gamma_0 + \theta\sigma^2 \tag{12}$$

If we repeat this procedure and multiply (9) by $y_{t-\tau}$ for $\tau \geq 2$, we find the following recursion

$$\gamma_\tau = \phi\gamma_{\tau-1} \quad \text{for } \tau \geq 2 \tag{13}$$

To find the variance of the process, replace (12) into (11) and isolate γ_0

$$\gamma_0 = \sigma^2\left(\frac{1+2\phi\theta+\theta^2}{1-\phi^2}\right)$$

Being defined γ_0, the autocorrelations $\rho_\tau = \frac{\gamma_\tau}{\gamma_0}$ can be computed from (12) and (13). Hence

$$\rho_1 = \phi + \frac{\theta\sigma^2}{\gamma_0} = \frac{(\phi+\theta)(1+\phi\theta)}{1+2\phi\theta+\theta^2}$$
$$\rho_\tau = \phi\rho_{\tau-1} \quad \text{for } \tau \geq 2$$

Part (iii) GARCH in mean. We will write down the log-likelihood function of the following model

$$y_t = \delta\sigma_t + u_t$$
$$u_t = \sigma_t\epsilon_t \quad \epsilon_t \sim NID(0,1)$$
$$\sigma_t^2 = \gamma + \alpha u_{t-1}^2 + \beta\sigma_{t-1}^2$$

Since y_t is not independent from past observations, the joint probability distribution equals $L(Y_T) = \Pi_{t=2}^T p(y_t|\mathbf{Y}_{t-1}) \cdot p(y_1)$ where $\mathbf{Y}_t = \{y_t, y_{t-1}, \ldots, y_2, y_1\}$. We have that $E_{t-1}[y_t] = \delta\sigma_t + E_{t-1}[u_t] = \delta\sigma_t$ and that $E_{t-1}[(y_t - E_{t-1}|y_t)^2] = E_{t-1}[u_t^2] = \sigma_t^2$, where $E_{t-1}[z] = E[z|\mathbf{Y}_{t-1}]$. Thus

$$y_t|\mathbf{Y}_{t-1} \sim N\left(\delta\sqrt{\gamma + \alpha u_{t-1}^2 + \beta\sigma_{t-1}^2}, \gamma + \alpha u_{t-1}^2 + \beta\sigma_{t-1}^2\right) \tag{14}$$

It is worth noting that if we assume that $u_0 = \sigma_0 = 0$, (14) is also a valid expression for y_1 and $y_1 \sim N(\delta\sqrt{\gamma}, \gamma)$. The log-likelihood function is

$$logL(\alpha,\beta,\delta,\gamma) = -\frac{T}{2}ln(2\pi) - \frac{1}{2}\sum_{t=1}^{T} ln(\tilde{\sigma}_t^2) - \frac{1}{2}\sum_{t=1}^{T}\left(\frac{y_t - \delta\tilde{\sigma}_t}{\tilde{\sigma}_t}\right)^2$$

where $\tilde{\sigma}_t^2 = \gamma + \alpha(y_{t-1} - \delta\tilde{\sigma}_{t-1})^2 + \beta\tilde{\sigma}_{t-1}^2$ with $y_0 = \tilde{\sigma}_0 = 0$.

Markov-Switching & Stochastic Volatility Models

Exercise 3 (Optional).

1. Why might we want to consider employing Markov-Switching models? If one was to entertain the possibility of using such models, briefly describe the basic setup, how optimal inference and forecasting might be conducted in a recursive manner, how we might start the algorithm and how we might estimate the parameters.

2. Why might we want to consider employing Stochastic-Volatility models? If one was to entertain the possibility of using such models, briefly describe the basic setup, a few properties of the model and some issues with the model.

Solution 3 (Markov-Switching & Stochastic Volatility models).

1. See section 6.3 of the notes in addition to Hamilton (1994) chapter 22.4, photocopies of which are handed out in class.

2. See section 6.4 of the notes.

Also see sections 1 and 2 plus pages 10–12 of Fernández-Villaverde & Rubio-Ramírez (2010) available at `economics.sas.upenn.edu/~jesusfv/macrovolatilityformat.pdf`. These sections are particularly informative on the overall trend for macro and volatility, especially the history and comparing and contrasting GARCH, Markov-Switching and Stochastic Volatility models. Finally, have a look at Hamilton, J. (2008) 'Macroeconomics and ARCH' available at `dss.ucsd.edu/~jhamilto/JHamilton_Engle.pdf`

Filtering

State Space Form & Kalman Filtering

Exercise 4 (20 Marks)**.** Consider the following MA(2) model

$$y_t = \epsilon_t + \theta_1 \epsilon_{t-1} + \epsilon_{t-2}$$

Write this model in state space form. (4 Marks) Is the model stationary? (1 Mark) How would you initialise the Kalman filter? (1 Mark) Write down the initial state vector $\mathbf{a}_0$, the initial MSE matrix $\mathbf{P}_0$, the first prediction error v_1 and its MSE f_1. (4 Marks) What would the updated state vector $\mathbf{a}_1$ and covariance matrix $\mathbf{P}_1$ be? (2 Marks) Write the prediction equations $\mathbf{a}_{2|1}$ and $\mathbf{P}_{2|1}$ and v_2 and f_2. (8 Marks)
Optional: derive the recursion for the Kalman filter in this case for v_t and f_t.

Solution 4 (State Space Form & Kalman Filter for MA(2))**.**
To put this model in state space form, define the state vector $\boldsymbol{\alpha}_t = (y_t, \epsilon_t, \epsilon_{t-1})$ and write

$$y_t = (1 \;\; 0 \;\; 0)\boldsymbol{\alpha}_t \quad t = 1, \ldots, T \tag{15}$$

$$\boldsymbol{\alpha}_t = \begin{bmatrix} 0 & \theta_1 & \theta_2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \boldsymbol{\alpha}_{t-1} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \epsilon_t \tag{16}$$

where (15) is the measurement equation and (16) is the state equation.

MA models are always stationary. So, the Kalman filter can be initialised with the mean and covariance matrix of the unconditional distribution of $\boldsymbol{\alpha}_t$ since $\boldsymbol{\alpha}_t$ is stationary. The initial state vector is $\mathbf{a}_0 = \mathbf{a}_{1|0} = \mathbf{0}$ and as $\boldsymbol{\alpha}_t = (y_t \epsilon_t, \epsilon_{t-1})'$, the initial covariance matrix $\mathbf{P}_0 = \mathbf{P}_{1|0}$ is

$$\mathbf{P}_{1|0} = \mathbf{P}_0 = \frac{1}{\sigma^2} E(\boldsymbol{\alpha}_t \boldsymbol{\alpha}_t') = \begin{bmatrix} 1 + \theta_1^2 + \theta_2^2 & 1 & \theta_1 \\ 1 & 1 & 0 \\ \theta_1 & 0 & 1 \end{bmatrix}$$

since we are dealing with an MA(2) process. The first prediction error is $v_1 = y_1$ and $f_1 = 1 + \theta_1^2 + \theta_2^2$. As here $Z_t = (1 \;\; 0 \;\; 0)$, updating we get that

$$\mathbf{a_1} = \begin{pmatrix} y_1 \\ \frac{y_1}{1+\theta_1^2+\theta_2^2} \\ \frac{\theta_1 y_1}{1+\theta_1^2+\theta_2^2} \end{pmatrix}$$

and

$$\mathbf{P}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{\theta_1^2+\theta_2^2}{1+\theta_1^2+\theta_2^2} & \frac{-\theta_1}{1+\theta_1^2+\theta_2^2} \\ 0 & \frac{-\theta_1}{1+\theta_1^2+\theta_2^2} & \frac{1+\theta_2^2}{1+\theta_1^2+\theta_2^2} \end{pmatrix}$$

We get the following prediction equations for $\boldsymbol{\alpha}_2$:

$$\begin{aligned}
\mathbf{a}_{2|1} &= \begin{pmatrix} \frac{\theta_1 y_1(1+\theta_2)}{1+\theta_1^2+\theta_2^2} \\ 0 \\ \frac{y_1}{1+\theta_1^2+\theta_2^2} \end{pmatrix} \\
\mathbf{P}_{2|1} &= \begin{bmatrix} \frac{1+\theta_1^2+\theta_2^2+\theta_1[\theta_1(\theta_1^2+\theta_2^2)-\theta_1\theta_2]+\theta_2[\theta_2(1+\theta_2^2)-\theta_1^2]}{1+\theta_1^2+\theta_2^2} & 1 & \frac{\theta_1(\theta_1^2+\theta_2^2)-\theta_1\theta_2}{1+\theta_1^2+\theta_2^2} \\ 1 & 1 & 0 \\ \frac{\theta_1(\theta_1^2+\theta_2^2)-\theta_1\theta_2}{1+\theta_1^2+\theta_2^2} & 0 & \frac{\theta_1^2+\theta_2^2}{1+\theta_1^2+\theta_2^2} \end{bmatrix} \\
v_2 &= y_2 - \frac{\theta_1 y_1(1+\theta_2)}{1+\theta_1^2+\theta_2^2} \\
f_2 &= \frac{1+\theta_1^2+\theta_2^2+\theta_1[\theta_1(\theta_1^2+\theta_2^2)-\theta_1\theta_2]+\theta_2[\theta_2(1+\theta_2^2)-\theta_1^2]}{1+\theta_1^2+\theta_2^2} \\
&= \frac{(1+\theta_1^2+\theta_2^2)^2-\theta_1^2(1+\theta_2)^2}{1+\theta_1^2+\theta_2^2}
\end{aligned}$$

Optional: further repetition reveals that the Kalman filter computes prediction erros from a recursion similar to the MA(1) process. The algebra is simple though extremely tedious so we leave it out. You will *not* be asked for such a general recursion under exam conditions.

ACGF & Spectrums

Exercise 5 (4 Marks)**.** Derive the autocovariance generating function and the spectral density function for the MA(1) process

$$y_t = \epsilon_t + \theta\epsilon_{t-1}$$

Hint: the MA(1) process may be expressed as $\Phi(L)y_t = \Theta(L)\epsilon_t$ where the lag polynomials are $\Theta(e^{i\omega}) = 1+e^{i\omega}$ and $\Phi(e^{i\omega}) = 1$.

Solution 5 (Autocovariance & Spectrum for MA(1))**.**
Consider the MA(1) process

$$y_t = \epsilon_t + \theta\epsilon_{t-1}$$

We can derive the autocovariance generating function as follows. First note that the autocovariance function will be

$$\gamma_\tau = \begin{cases} (1+\theta^2)\sigma^2 & \tau = 0 \\ \theta\sigma^2 & \tau = 1 \\ 0 & \tau > 1 \vee \tau < -1 \end{cases}$$

You should ensure you can derive these. Now looking at the autocovariance generating function

$$\begin{aligned}
f_Y(\omega) &= \sum_{\tau=-\infty}^{\infty} \gamma(\tau)e^{-i\omega\tau} \\
&= (1+\theta^2)\sigma^2 + \theta\sigma^2(e^{-i\omega}+e^{i\omega}) \\
&= \sigma^2[1+\theta(e^{-i\omega}+e^{i\omega})+\theta^2] \qquad (17)
\end{aligned}$$

Note that the hint would have allowed us to do this faster by observing that from example 7.38 in the notes, the autocovariance generating function for a general ARMA(p,q) is given by

$$f_Y(\omega) = \frac{\sigma^2\Theta(e^{i\omega})\Theta(e^{-i\omega})}{\Phi(e^{i\omega})\Phi(e^{-i\omega})}$$

and substituting the lag polynomials for the MA(1) yields (17).

The spectral density function is

$$\begin{aligned} s_Y(\omega) &= \frac{1}{2\pi} f_Y(\omega) \\ &= \frac{\sigma^2}{2\pi}[1 + \theta(e^{-i\omega} + e^{i\omega}) + \theta^2] \\ &= \frac{\sigma^2}{2\pi}[1 + 2\theta\cos(\omega) + \theta^2] \end{aligned}$$

where the last line follows from observing that $e^{-i\omega} + e^{i\omega} = 2\cos(\omega)$.

Seasonal difference filter

Exercise 6 (2 Marks)**.** Derive the gain function for the seasonal difference filter, $h(L) = 1 - L^{12}$.

Solution 6 (Gain of Seasonal Difference Filter)**.**
From definition 7.51 in the notes, the gain function is

$$G(\omega) = |C(e^{-i\omega})|$$

where $|C(e^{-i\omega})|$ is the modulus of $C(e^{-i\omega})$, the frequency response function, i.e.

$$|C(e^{-i\omega})| = \sqrt{C(e^{-i\omega})C(e^{i\omega})}$$

As the seasonal difference filter is $h(L) = 1 - L^{12}$,

$$\begin{aligned} G(\omega) &= \sqrt{(1 - e^{-i\omega 12})(1 - e^{i\omega 12})} \\ &= \sqrt{2}\sqrt{1 - \cos(12\omega)} \end{aligned}$$

where the second inequality follows from the fact that $e^{-i\omega} + e^{i\omega} = 2\cos(\omega)$.

Real time filtering & filtered data

Exercise 7 (5 Marks)**.**

1. Suppose you are asked to estimate the output gap in real time. How might you go about this and what issues might be present in using two-sided filters? (3 Marks)
2. Discuss some issues in conducting regressions using filtered data. (2 Marks)

Solution 7 (Two sided filters & real time data plus regression using filtered data)**.**

1. On the first question, see the introductory paragraph in section 7.2.4 and the paragraph starting from the bottom of page 201 onwards until I discuss regressions using filtered data.
2. On the second question, see my discussion of regression using filtered data starting from page 202 and ending on the next page (up to the end of section 7.2.4).

Hodrick-Prescott filter

Exercise 8 (7 Marks)**.** The gain function of the Hodrick-Prescott filter is given by

$$G(\omega) = \left[1 + \left(\frac{\sin(\omega/2)}{\sin(\omega_0/2)}\right)^4\right]^{-1}$$

where

$$\omega_0 = 2arcsin\left(\frac{1}{2\lambda^{\frac{1}{4}}}\right)$$

Different specifications of λ imply that 50% of the filter gain has been completed at a particular number of cycles, e.g. 40-quarter cycle.

Using MATLAB, determine the values of λ for which 50% of the filter gain has been completed at

1. 40-quarter cycle (10 years for quarterly data) (1 Mark)
2. 10-year cycle (annual data) (1 Mark)
3. 120-month cycle (monthly data) (1 Mark)
4. 32-quarter cycle (1 Mark)
5. 6-quarter cycle (1 Mark)
6. 20-quarter cycle (1 Mark)
7. 56-quarter cycle (1 Mark)

Solution 8 (Hodrick-Prescott Filter)**.**
When 50% of the filter gain has been completed, we can solve for λ for different data frequencies, starting by equating

$$\begin{aligned}
\frac{1}{2} &= \left[1 + \left(\frac{\sin\left(\frac{\omega}{2}\right)}{\sin\left(\frac{\omega_0}{2}\right)}\right)^4\right]^{-1} \\
\iff 2 &= 1 + \left(\frac{\sin\left(\frac{\omega}{2}\right)}{\sin\left(\frac{\omega_0}{2}\right)}\right)^4 \\
\iff 1 &= \frac{\sin\left(\frac{\omega}{2}\right)}{\sin\left(\frac{\omega_0}{2}\right)}
\end{aligned}$$

Plugging in the expression for ω_0, we get that

$$\begin{aligned}
1 &= \frac{\sin\left(\frac{\omega}{2}\right)}{\sin\left(\arcsin\left(\frac{1}{2\lambda^{\frac{1}{4}}}\right)\right)} \\
\iff \sin\left(\frac{\omega}{2}\right) &= \sin\left(\arcsin\left(\frac{1}{2\lambda^{\frac{1}{2}}}\right)\right) = \frac{1}{2\lambda^{\frac{1}{4}}} \\
\iff \lambda &= \left[\frac{1}{2\sin\left(\frac{\omega}{2}\right)}\right]^4
\end{aligned}$$

Plugging in the values of ω, we get the corresponding values of λ as shown in table 1. The first three entries show the typical calibration for λ in the Hodrick-Prescott filter for quarterly, annual and monthly data, respectively. These corresponds to the business cycle frequency. However, when we increase the frequency (reduce the period) to say 32 quarters, 20 quarters or even 6 quarters (all with quarterly data), notice how we must reduce λ even though we are still using quarterly data – this is because we are focusing on higher-frequency data. Similarly, for lower frequency (longer period) quarterly data, e.g. 56 quarters, we must increase λ even though we are still using quarterly data – this is because we are focusing on low-frequency data. While the first three parts provide you with the explanation why the standard numbers are chosen for HP filters (quarterly/annual/monthly – *business cycle frequency*), the latter parts show you why you still

ω	λ
40 Quarters	$\approx 1,650$
10 Years	≈ 6.85
120 Months	$\approx 133,107$
32 Quarters	≈ 677
6 Quarters	≈ 1
20 Quarters	≈ 104
56 Quarters	$\approx 6,383$

Table 1: Values of λ for Hodrick Prescott filter for different data frequencies.

must make adjustments if you want to focus on different frequencies/periods, holding the data type constant (e.g. quarterly data).[1]

[1] This can get confusing. Lower frequency data is often synonymously referred to as annual data and higher frequency data is often synonymously referred to as quarterly or monthly data. What I mean by frequency above is really a function of the period you are interested in studying, i.e. long run (low frequency), business cycle (medium frequency) or extremely short run (high frequency) such as seasonal variation, etc.